Index of Coregularity 0 log Calabi–Yau Pairs (Filipazzi–Mauri–Moraga)

Notations. A generalized log Calabi–Yau pair (X,B,M): (g-log CY pair)

- (X,B,M) has generalized lc singularities.
- $K_X+B+M_X\sim_{\mathbb{Q}}0$

Weil index of $(X,B,M)=\min\{\lambda\in\mathbb{Z}_+:\lambda(K_X+B+M_X)$ is integral$\}$

(Cartier) index of $(X,B,M)=\min\{\lambda\in\mathbb{Z}_+:\lambda(K_X+B+M_X)$ is Cartier$\}$.

Index Conj.:

Fix n and a DCC set $I\subseteq\mathbb{Q}_+\cap[0,1]$. Then $\exists\, N=N(n,I)$ s.t.

for any log CY pair (X,B) satisfying $\dim X=n$, $\mathrm{coeff}(B)\subseteq I$,

we have $N(K_X+B)\sim 0$.

Known results:

- For $n=2$, $I=\{1-\frac{1}{m}\}$, one can take $N=66$. (Mukai, Ishii, ... '90s)

  The bound $N=66$ sharp (Kondo '92).
- For canonical CY 3-fold X: Kawamata–Morrison '86.
- For lc CY 3-fold: C. Jiang ('20). Y. Xu ('20).
- Not known for $n\geq 4$. (in general).

Observations: numerical invariants such as lct and index should only depend on coregularity, but not on dim.

Def For a dlt pair (X, B), dual complex $\mathcal{D}(X,B)$ is constructed:

$$B^{=1} = B_1 + \cdots + B_r \quad (\mathrm{SNC}).$$

Vertices in $\mathcal{D}(X,B)$ $\longleftrightarrow$ Divisors B_i.

k-dim simplices $\longleftrightarrow$ Components of $B_{j_0} \cap \cdots \cap B_{j_k}$.

Gluing $\longleftrightarrow$ Inclusions.

PL-homeo class of polyhedral complexes.

For a glc pair (X, B, M), define $\mathcal{D}(X,B,M)$ to be (an equiv class of) $\mathcal{D}(Y, B_Y + M_Y)$, where (Y, B_Y, M) is a gdlt modification of (X, B, M).

The coregularity of $(X,B,M) := \dim X - 1 - \dim \mathcal{D}(X,B,M)$

In particular, coreg $0 \Leftrightarrow \exists$ 0-dim glc center

$\Leftrightarrow$ Some strata of $B_Y^{=1}$ is a point.

Thm A (FMM) Let (X, B, M) be a proj g-log CY pair of coreg 0 and Weil index λ.

Then $\lambda'(K_X + B + M_X) \sim 0$, where $\lambda' = \mathrm{lcm}(\lambda, \underline{\underline{2}})$.

explain where this 2 comes from.

Cor Let (X, B) be a proj log CY pair of coreg 0 with $B = B^{=1}$.

Then $2(K_X + B) \sim 0$.

Rmk Same result holds for slc log CY pairs

or for case $\mathrm{coeff}(B) \geq \frac{1}{2}$.

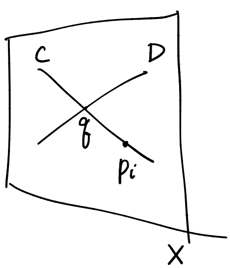
Thm B (FMM) Let (X, B) be a proj dlt log CY pair of coreg 0, dim n, $B = B^{=1}$.

Then $H^{n-1}_{\mathrm{sing}}(\mathcal{D}(X, B); \mathbb{C}) \cong H^0(X, K_X + B)$.

(1) $K_X + B \sim 0$ iff $\mathcal{D}(X, B)$ is "orientable".

(2) If $\mathcal{D}(X, B)$ is not orientable, then $\exists$ quasi-étale double cover $(\tilde{X}, \tilde{B}) \to (X, B)$ such that $\mathcal{D}(\tilde{X}, \tilde{B})$ is orientable and $\mathcal{D}(X, B) \cong \mathcal{D}(\tilde{X}, \tilde{B}) / (\mathbb{Z}/2)$.

Example Let (X, B) be a dlt log CY surface with $B = C + D$, $C \cap D = \{q\}$.



Suppose $p_1, \ldots, p_r$ cyclic quot sing on X which lie on C.

Then by adjunction,

$$0 \sim_{\mathbb{Q}} (K_X + B)|_C = K_C + D|_C + \mathrm{Diff}_C(0).$$

$$= K_C + q + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) p_i. \quad m_i = \text{order of cyclic quotient}$$

$$\Rightarrow 0 = 2g(C) - 2 + 1 + \sum_i \left(1 - \frac{1}{m_i}\right)$$

$$\Rightarrow g(C) = 0, \quad r = 2, \quad m_1 = m_2 = 2.$$

Note: $K_X + B$ has index 1, but $(K_X + B)|_C$ has index 2.

Properties of indices and coregs.

(X, B, M) g-log CY pair of coreg 0.

1. (invariance under crepant transformations)

   Suppose (X, B, M), (X', B', M') are crepant bir'l, then they have the same index and coreg.

2. (invariance under adjunction).

   Let λ = Weil index of (X, B, M). S = (normalized) prime comp. of $B^{=1}$.

   $$K_S + B_S + N_S \sim_{\mathbb{Q}} (K_X + B + M_X)|_S.$$

   Then (S, B_S, N) is g-log CY pair of coreg 0 and its Weil index divides $\lambda' = \mathrm{lcm}(\lambda, 2)$.

3. Suppose M_X is $\mathbb{Q}$-Cart. and $\equiv 0$, let c = b-Weil index of M_X

   Then $cM \sim 0$ as b-divisors.

4. Coreg is preserved under finite quot.

Strategy of Pf of Thm A.

By Property 1, can assume (X, B, M) is $\mathbb{Q}$-fact gdlt.

Case 1: $B^{<1} + M_X \equiv 0$.

$$\Rightarrow B^{<1} = 0, \ M_X \equiv 0. \overset{\text{(Property 3)}}{\Longrightarrow} \lambda \cdot M_X \sim 0.$$

This reduces to the case of a pair $(X, B = B^{=1})$. (exactly Cor)

Case 2: $B^{<1} + M_X \not\equiv 0. \Rightarrow K_X + B^{=1}$ is not pseff.

Run

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Case 2: $B^{<1} + M_X \not\equiv 0$. $\Rightarrow K_X + B^{=1}$ is not pseff.

Run $(K_X + B^{=1})$-MMP to get MFS $g: Y \to Z$.

By Property 1, can reduce to $\lambda'(K_Y + B_Y + M_Y) \sim 0$.

Pick $S \subseteq B_Y^{=1}$, and by adjunction

$$K_S + B_S + N_S \sim_{\mathbb{Q}} (K_Y + B_Y + M_Y)|_S.$$

(use Property 2)

$\lambda'(K_S + B_S + N_S)$ is integral, $\mathbb{Q}$-trivial $\overset{\text{by induction}}{\Longrightarrow}$ $\lambda'(K_S + B_S + N_S) \sim 0$.

Finally, use KV vanishing to lift the trivializing section on S back to a trivializing section on Y.

Rmk Coreg 0: Z cannot be higher dim. (b/c all the $B^{=1}$ would dominate Z).

$X \longrightarrow Z$ MRC fibration.

Fibers are RC.

Orientability of dual complex.

Fact: $\mathcal{D}(X,B)$ is a $(n-1)$-dim pseudo-manifold (up to PL-homeomorphism), i.e., a polyhedral complex which is

- pure-dim of dim $n-1$
- no branches
- connected.

Non-examples of ps-mflds:

Def / Lemma Let T be a ps-mfld of dim n.

Then $H^n(T;\mathbb{Z}) = \mathbb{Z}$ or 0.

We say T is orientable if $H^n(T;\mathbb{Z}) = \mathbb{Z}$.

Prop. Let (X,B) proj dlt pair of coreg 0 and $B = B^{=1}$.

Then $\mathcal{D}(X,B)$ is orientable $\iff K_X + B \sim 0$.

Sketch $\dim X = n$. Consider

$$\cdots \longrightarrow \overset{=0}{H^{n-1}(X,\mathcal{O}_X)} \longrightarrow H^{n-1}(B,\mathcal{O}_B) \longrightarrow H^n(X,\mathcal{O}_X(-B)) \longrightarrow \overset{=0}{H^n(X,\mathcal{O}_X)} \longrightarrow \cdots$$

(X,B) has coreg 0 $\Rightarrow$ X is rationally connected (Kollár–Xu '16).

$\Rightarrow H^i(X,\mathcal{O}_X) = 0,\ \forall i>0$ (KMM '92)

Thus, $H^{n-1}(B,\mathcal{O}_B) \cong H^n(X,\mathcal{O}_X(-B)) \cong H^0(X, K_X+B)$.

Claim: (Friedman–Morrison '83).

$$H^{n-1}(B,\mathcal{O}_B) \cong H^{n-1}(\mathcal{D}(X,B);\mathbb{C}).$$

Sketch pf of claim: (Kollár)

$$B = B_1 + \cdots + B_r$$

$$0 \to \mathcal{O}_B \to \sum_i \mathcal{O}_{B_i} \to \sum_{i<j} \mathcal{O}_{B_i \cap B_j} \to \sum_{i<j<k} \mathcal{O}_{B_i \cap B_j \cap B_k} \to \cdots$$

This gives a spectral seq:

$$\sum_{|J|=q} H^p(B_J, \mathcal{O}_{B_J}) \Longrightarrow H^{p+q}(B, \mathcal{O}_B). \qquad J \subseteq \{1, \dots, r\},$$
$$B_J = \bigcap_{j \in J} B_j$$

Since $(B_J, (B - \sum_{j \in J} B_j)|_{B_J})$ has coreg 0,

$$B_J \text{ is RC} \Rightarrow H^i(B_J, \mathcal{O}_{B_J}) = 0 \quad \forall i > 0.$$

Thus, spectral seq concentrated along $p=0$, so it degenerates after the next page.

$\Rightarrow$ $H^*(B, \mathcal{O}_B)$ is the cohomology of $p=0$ line:

$$\cdots \to \sum_{|J|=p} H^0(B_J, \mathcal{O}_{B_J}) \to \sum_{|J|=p+1} H^0(B_J, \mathcal{O}_{B_J}) \to \cdots$$
$$\wr\wr$$
$$\cdots \to \sum_{|J|=p} \mathbb{C}^{\#(p-1)\text{-simplices in } \mathcal{D}(X,B)} \to \cdots$$

This is cellular cochain complex for $\mathcal{D}(X,B)$.

Thus $H^*(B, \mathcal{O}_B) \cong H^*_{\mathrm{sing}}(\mathcal{D}(X,B); \mathbb{C})$. $\square$

Now we know

$$H^{n-1}(\mathcal{D}(X,B);\mathbb{C}) \cong H^0(X, K_X+B).$$

Thus, $\mathcal{D}(X,B)$ is orientable

$$\Leftrightarrow \quad H^0(X, K_X+B) \cong \mathbb{C}.$$

$$\Leftrightarrow \quad K_X+B \sim 0. \qquad \square.$$

<u>Sketch Pf of Thm B.</u> part (2).

Suppose index of K_X+B is m.

Let $q: (Y, B_Y) \to (X,B)$ be index 1 cover.

Then q is quasi-étale and deg m. (Y, B_Y) dlt log CY of coreg 0. $K_Y+B_Y \sim 0$.

Galois group $\mathbb{Z}/m$ acts on $\mathcal{D}(Y, B_Y)$. and

$$\mathcal{D}(Y, B_Y) / (\mathbb{Z}/m) \cong \mathcal{D}(X,B).$$

By Prop., $H^{n-1}(\mathcal{D}(Y, B_Y);\mathbb{Q}) \cong \mathbb{Q}$

$$H^{n-1}(\mathcal{D}(X, B);\mathbb{Q}) = 0.$$

But $H^{n-1}(\mathcal{D}(X,B);\mathbb{Q}) \cong H^{n-1}(\mathcal{D}(Y, B_Y);\mathbb{Q})^{\mathbb{Z}/m} = 0.$

$\Rightarrow$ $\mathbb{Z}/m$ acts nontrivially on $H^{n-1}(\mathcal{D}(Y, B_Y);\mathbb{Q}) \cong \mathbb{Q}$.

$\Rightarrow$ $\mathbb{Z}/m$ acts as ± 1 on $\mathbb{Q}$.

$\Rightarrow$ m is even.

Consider $(\tilde{X}, \tilde{B})$ be the quot. of (Y, B_Y) by $\mathbb{Z}/\frac{m}{2}$

$$(Y, B_Y) \xrightarrow{\frac{m}{2}:1} (\tilde{X}, \tilde{B}) \xrightarrow{2:1} (X, B)$$

$$\mathcal{D}(Y, B_Y) \xrightarrow{\frac{m}{2}:1} \mathcal{D}(\tilde{X}, \tilde{B}) \xrightarrow{2:1} \mathcal{D}(X, B)$$

(Property 4) $(\tilde{X}, \tilde{B})$ is a dlt log CY pair of coreg 0.

such that $\mathcal{D}(\tilde{X}, \tilde{B})$ is orientable.

$\left(H^{n-1}(\mathcal{D}(\tilde{X}, \tilde{B}); \mathbb{Q}) = H^{n-1}(\mathcal{D}(Y, B_Y); \mathbb{Q})^{\mathbb{Z}/\frac{m}{2}} = \mathbb{Q} \right)$

and $\mathcal{D}(\tilde{X}, \tilde{B}) / (\mathbb{Z}/2) \cong \mathcal{D}(X, B)$.

We also get $K_{\tilde{X}} + \tilde{B} \sim 0$.

Let $\tilde{f}$ s.t. $K_{\tilde{X}} + \tilde{B} = \operatorname{div}(\tilde{f})$.

Then $\operatorname{tr}(\tilde{f}) \in \mathcal{O}_X$ is a trivializing section of

$$\operatorname{tr}(K_{\tilde{X}} + \tilde{B}) = 2(K_X + B).$$

$\Rightarrow 2(K_X + B) \sim 0$. Hence $m = 2$. $\blacksquare$